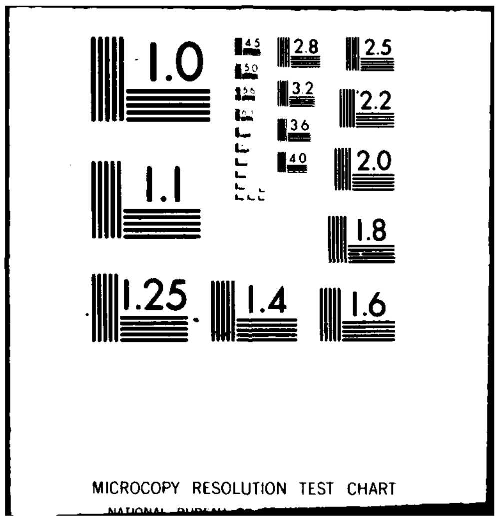
AD-A088 182 SOUTH CAROLINA UNIV COLUMBIA DEPT OF MATHEMATICS COM--ETC F/G 12/1
NONPARAMETRIC EMPIRICAL BAYES ESTIMATION OF RELIABILITY.(U)
MAY 80 K Y LIANG, W J PADGETT F49620-79-C-0140
UNCLASSIFIED TR-58 AFOSR-TR-80-0587 NL

END
DATE
FILMED
9-80
DTIC


1.0
4.5
5.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6
MICROCOPY RESOLUTION TEST CHART

AFOSR-TR- 80-0587

LEVEL II

AD A088182

NONPARAMETRIC EMPIRICAL BAYES
ESTIMATION OF RELIABILITY

by

K. Y. Liang and W. J. Padgett*
University of South Carolina
Statistics Technical Report No. 58
62G05-3

May 1980

DTIC
ELECTE
AUG 19 1980

Department of Mathematics, Computer Science, and Statistics
University of South Carolina
Columbia, South Carolina 29208

* This author supported by the ~~United States Air~~ Force Office of Scientific Research under Contract No. F49620-79-C-0140

80 8 14 081

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

| | | |
|---|---|---|
| 1. REPORT NUMBER<br>**AFOSR-TR- 80-0587** | 2. GOVT ACCESSION NO.<br>AD-A088 182 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>Nonparametric Empirical Bayes Estimation of Reliability | | 5. TYPE OF REPORT & PERIOD COVERED<br>Interim<br>6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>K. Y. Liang and W. J. Padgett | | 8. CONTRACT OR GRANT NUMBER(s)<br>F49620-79-C-0140 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>University of South Carolina, Department of Mathematics, Computer Science, and Statistics<br>Columbia, South Carolina 29208 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>61102F 2304/A5 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Air Force Office of Scientific Research/NM<br>Bolling AFB, Washington, D.C. 20332 | | 12. REPORT DATE<br>May, 1980<br>13. NUMBER OF PAGES<br>12 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>UNCLASSIFIED<br>15a. DECLASSIFICATION DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Empirical Bayes; Nonparametric estimation; Asymptotic optimality; Distribution function; Dirichlet process prior.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Several sequences of nonparametric empirical Bayes estimators of a distribution (or reliability) function are considered. Their asymptotic optimality relative to a Dirichlet process prior is investigated, and the estimators are compared for a small number of stages with respect to Weibull distributions by computer simulation.

DD FORM 1473 EDITION OF 1 NOV 65 ...

Unclassified

## Summary

Several sequences of nonparametric empirical Bayes estimators of a distribution (or reliability) function are considered. Their asymptotic optimality relative to a Dirichlet process prior is investigated, and the estimators are compared for a small number of stages with respect to Weibull distributions by computer simulation.

## 1. Introduction

In this note we consider a nonparametric approach to estimating an unknown probability distribution function, or equivalently, a reliability function. That is, nothing is assumed to be known about the specific form or parameters of the distribution. Specifically, nonparametric empirical Bayes estimation will be considered in that a prior distribution over the space of all probability distributions is assumed to exist but is not completely specified. Korwar and Hollander (1976, 1977) have taken such an approach based on the nonparametric Bayes estimation of a distribution function given by Ferguson (1973, 1974). We will present two additional nonparametric empirical Bayes estimators of a distribution function, examine their properties, and compare them with the Korwar-Hollander estimators. These estimators appear to be plausible alternatives to the Korwar-Hollander estimators.

Let $(P_i, \underline{X}_i)$, $i = 1, 2, \ldots$, be a sequence of independent random elements, where $P_i$ are random probability measures on the real line and, given $P_i = P$, $\underline{X}_i = (X_{i1}, \ldots, X_{im_i})$ is a random sample from P. Let $F_i$ denote the corresponding random distribution function for each $P_i$, $i = 1, 2, \ldots$ . The $P_i$ are taken to have a common prior distribution given by a Dirichlet process on the measurable space $(R, \mathcal{B})$, where R denotes the real line and $\mathcal{B}$ is the $\sigma$-field of Borel subsets of R. The parameter of the Dirichlet process will be denoted by $\alpha(\cdot)$, a $\sigma$- additive finite nonnull measure on $(R, \mathcal{B})$. (See Ferguson's (1973, 1974) papers for basic definitions and properties of Dirichlet processes.)

We consider the problem of estimating the distribution function $F_{n+1}(t) = P_{n+1}((-\infty, t])$ in this empirical Bayes framework with respect to the

loss function $L(F, F^*) = \int_R [F(t) - F^*(t)]^2 \, dW(t)$, where $W(t)$ is a specified nonrandom weight function and $F^*$ is an estimator of F. Korwar, et al (1976, 1977) proposed the sequence of estimators

$$G_{n+1}(t) = p_{m_{n+1}} \sum_{i=1}^{n} \hat{F}_i(t)/n + (1-p_{m_{n+1}}) \hat{F}_{n+1}(t), \; n = 1, 2, \ldots, \tag{1.1}$$

where $p_{m_n} = \alpha(R)/[\alpha(R) + m_n]$. Exact risk expressions were obtained and the rate at which the overall expected loss for $G_{n+1}$ converged to the minimum Bayes risk (attained by Ferguson's (1973) nonparametric Bayes estimators) was indicated. Here two other sequences of estimators are proposed and their asymptotic optimality and comparison with (1.1) are considered.

## 2. The Estimators and Their Asymptotic Optimality

Let $M = \{M_{n+1}\}$ represent a sequence of estimators of an unknown distribution function F. In our empirical Bayes framework, Ferguson's (1973, p. 222) Bayes estimator of F based on the (n+1)st stage sample $\underline{X}_{n+1}$ is given by

$$\tilde{F}_{m_{n+1}}(t) = p_{m_{n+1}} F_0(t) + (1-p_{m_{n+1}}) \hat{F}_{n+1}(t), \tag{2.1}$$

where $F_0(t) = \alpha((-\infty, t])/\alpha(R)$ and $\hat{F}_{n+1}$ is the sample distribution function of $\underline{X}_{n+1}$. Then the Bayes risk $R_{n+1}(\alpha)$ of (2.1) is given by

$$R_{n+1}(\alpha) = E_{\underline{X}_{n+1}}\{\int [E_{F(t)|\underline{X}_{n+1}} (F(t) - \tilde{F}_{m_{n+1}}(t))^2] dW(t)\}, \tag{2.2}$$

and the risk of $M_{n+1}$ is

$$R(M_{n+1}, \alpha) = E_{\underline{X}_{n+1}}\{\int [E_{F(t)|\underline{X}_{n+1}} (F(t) - M_{n+1}(t))^2] dW(t)\}.$$

Denote the expectation of $R(M_{n+1}, \alpha)$ with respect to $\underline{X}_1, \ldots, \underline{X}_n$ by $R_{n+1}(M, \alpha)$.

Definition 2.1. The sequence $M = \{M_{n+1}\}$ is said to be asymptotically optimal relative to $\alpha$ if $R_{n+1}(M, \alpha)/R_{n+1}(\alpha) \to 1$ as $n \to \infty$.

We note that when the sample sizes at each stage n are equal, then Definition 2.1 reduces to that of Korwar et al (1976, Definition 2.3). In this case, $R_{n+1}(\alpha) = R(\alpha)$, the minimum Bayes risk for Ferguson's estimator.

For completeness we state Lemma 2.5 of Korwar et al (1976).

Lemma 2.1. Let P be a Dirichlet process on $(R, \mathcal{B})$ with parameter $\alpha$, and let $X_1, \ldots, X_m$ be a sample of size m from P with distribution function $F(t) = P((-\infty, t])$. Let $\hat{F}(t)$ be the sample distribution function of $\underline{X} = (X_1, \ldots, X_m)$. Then for each $t \in R$

$$E(F(t)|\underline{X}) = \tilde{F}_m(t),$$

$$E(F(t)) = F_0(t),$$

and

$$E(F^2(t)) = F_0(t)/m + (m-1)F_0(t)\{F_0(t)\alpha(R)+1\}/\{m(\alpha(R)+1)\},$$

where

$$\tilde{F}_m(t) = p_m F_0(t) + (1-p_m)\hat{F}(t) \quad \text{and} \quad p_m = \alpha(R)/[\alpha(R)+m].$$

Korwar et al (1977) proved the following theorem.

Theorem 2.1 Let $\alpha(R)$ be known. Then the sequence $G = \{G_{n+1}\}$ defined by (1.1) is asymptotically optimal relative to $\alpha$.

We now introduce two other sequences of estimators which seem to be natural candidates for empirical Bayes estimation. We discuss their asymptotic

risk behavior and in Section 3 consider some of their small sample properties and their behavior during early stages of the empirical Bayes estimation as compared with the sequence (1.1).

If the sample sizes at the various stages are equal, $m_n = m$, $n=1, 2, \ldots$, the estimator $G_{n+1}$ puts equal weights on each of the previous n sample distribution functions. In some situations, it might be desirable to place more weight on samples which occur at the most recent stages than those which are observed at the beginning of the process. A sequence of estimators which is appealing in this sense is defined by

$$G^*_{n+1}(t) = p_m G^*_n(t) + (1-p_m) \hat{F}_{n+1}(t), \; n=1, 2, \ldots \tag{2.3}$$

where $G^*_1(t) = \hat{F}_1(t)$. The next theorem shows that $G^* = \{G^*_{n+1}\}$ is not exactly asymptotically optimal relative to $\alpha$, but can be made $\varepsilon$-asymptotically optimal as discussed after the proof.

Theorem 2.2. As $n \to \infty$, $R_{n+1}(G^*,\alpha)$ converges to $[1 + \alpha(R)/(2\alpha(R)+m)]R(\alpha)$.

Proof. First, we write $G^*_{n+1}(t)$ as

$$G^*_{n+1}(t) = p^n_m \hat{F}_1(t) + p^{n-1}_m (1-p_m)\hat{F}_2(t) + \ldots$$
$$+ p_m(1-p_m)\hat{F}_n(t) + (1-p_m)\hat{F}_{n+1}(t).$$

Now, similar to Equation (2.12) of Korwar et al (1976), it can be shown that

$$R_{n+1}(G^*,\alpha) = R(\alpha) + \int E_{\underline{X}_1,\ldots,\underline{X}_n} (\tilde{F}_m(t) - G^*_{n+1}(t))^2 dW(t).$$

After some straightforward algebra and applying Lemma 2.1, it is easy to show that as $n \to \infty$

$$(2.4) \qquad R_{n+1}(G^*,\alpha) \to R(\alpha) + [\alpha^2(R)/(\alpha(R)+m)(2\alpha(R)+m)(\alpha(R)+1)] \times \int F_0(t)(1-F_0(t))dW(t).$$

However, according to Equation (2.19) of Korwar et al (1976),

$$R(\alpha) = [\alpha(R)/(\alpha(R)+1)(\alpha(R)+m)]\int F_0(t)(1-F_0(t))dW(t).$$

Thus, after simplification, (2.4) becomes

$$R_{n+1}(G^*,\alpha) \to (1 + \alpha(R)/(2\alpha(R)+m))R(\alpha). \qquad ///$$

Note that if we increase the sample size m, the difference between $\lim_{n\to\infty} R_{n+1}(G^*,\alpha)$ and $R(\alpha)$ will become smaller, and we can call $\{G^*_{n+1}\}$ $\varepsilon$-<u>asymptotically optimal relative to</u> $\alpha$ in this case, since for any $\varepsilon > 0$ we can choose m so that $\lim_{n\to\infty} R_{n+1}(G^*,\alpha)$ is within $\varepsilon$ of $R(\alpha)$.

The second sequence of estimators which we consider is defined by

$$(2.5) \qquad H_{n+1}(t) = p_{m_{n+1}}\hat{S}_n(t) + (1-p_{m_{n+1}})\hat{F}_{n+1}(t), \; n=1,2,\ldots,$$

where $\hat{S}_i$ is the sample distribution function of the <u>pooled</u> observations $\underline{X}_1,\ldots,\underline{X}_i$. Note that $H_{n+1}(t)$ is exactly the same as $G_{n+1}(t)$ when $m_n = m$ for each n. However, the asymptotic optimality of $\{H_{n+1}\}$ for the case that the sample sizes are not constant requires a restriction on the sample sizes at each step as the next theorem shows. This condition results from the fact that the pooled sample from which $\hat{S}_n$ is obtained is of size $K_n = \sum_{i=1}^n m_i$.

<u>Theorem 2.3</u>. For unequal sample sizes, the sequence of estimators $H = \{H_{n+1}\}$ is asymptotically optimal relative to $\alpha$ if and only if $m_n \to \infty$ as $n \to \infty$.

Proof: Let $K_n = \sum_{i=1}^{n} m_i$. Similar to the proof of Theorem 2.2, we have

$$R_{n+1}(H,\alpha) = R_{n+1}(\alpha) + \int E_{\underline{X}_1,\ldots,\underline{X}_n}(\tilde{F}_{m_{n+1}}(t)-H_{n+1}(t))^2 dW(t), \tag{2.6}$$

where

$$E_{\underline{X}_1,\ldots,\underline{X}_n}(\tilde{F}_{m_{n+1}}(t) - H_{n+1}(t))^2 = p_{m_{n+1}}\{F_0^2(t)-2F_0(t)E[\hat{S}_n(t)] + E[\hat{S}_n^2(t)]\}. \tag{2.7}$$

Applying Lemma 2.1 to the expectations on the right side of (2.7), equation (2.6) becomes

$$R_{n+1}(H,\alpha) = [1 + \alpha(R)(\alpha(R)+K_n)/K_n(\alpha(R)+m_{n+1})]R_{n+1}(\alpha). \tag{2.8}$$

Hence, $R_{n+1}(H,\alpha)/R_{n+1}(\alpha) \to 1$ as $m_n \to \infty$. ///

We can compare the performance of the estimator $H_{n+1}$ to that of the sample distribution function $\hat{F}_{n+1}$ at each stage. The following corollary to Theorem 2.3 shows that, under certain mild conditions on the sample sizes $m_{n+1}$, $H_{n+1}$ is better than the sample distribution function in the sense that $H_{n+1}$ has smaller overall expected loss.

Corollary 2.1. For each $n = 1,2,\ldots$, $R(\hat{F}_{n+1},\alpha) > R_{n+1}(H,\alpha)$ if and only if $K_n > m_{n+1}$.

Proof. From equation (3.3) of Korwar et al (1976),

$$R(\hat{F}_{n+1},\alpha) = [1 + \alpha(R)/m_{n+1}]R_{n+1}(\alpha). \tag{2.9}$$

Hence, comparing (2.8) and (2.9), the result follows. ///

We have considered the asymptotic optimality of the proposed sequences of

estimators of a distribution function in an empirical Bayes setting. In general, however, the comparison of the three sequences for small values of n by analytical methods is difficult, if not impossible. Monte Carlo simulations have been performed, assuming that $F_i$ is a Weibull distribution with a known shape parameter and random scale parameter $\beta$. Some of the results of the simulations are given in the next section.

## 3. Monte Carlo Comparisons

In this section, we implement Monte Carlo simulation of random lifetimes to study properties of and compare the empirical Bayes estimators discussed in Section 2.

The Weibull distribution $F(t) = 1 - \exp[t^{\gamma}/\beta]$, $(t \geq 0)$, was taken to be the failure model and was assumed to be the "correct" model reflecting past knowledge. With the parameter $\gamma$ fixed, we assume $\beta$ is randomly distributed with the exponential distribution as the prior distribution (Canavos and Tsokos (1973)).

For each fixed $\gamma, \alpha(R)$, and $\lambda$ (the parameter of the exponential prior distribution for $\beta$),the simulations were performed as follows:

1. Fifteen values of $\beta$ were generated from the assumed exponential prior distribution with parameter $\lambda$. The true reliability $R(t)$ for the Weibull distribution was computed and stored for each of the 15 stages, where t is chosen such that $R(t) \approx 0.4$.

2. A sample of size $m_n$ was generated from a Weibull distribution for each of the 15 values of $\beta$, representing 15 stages of the process. Three sequences of estimators were then computed according to (1.1), (2.3) and (2.5), and the squared error between those values and the true reliabilities were stored for each of 15 stages.

3. With the same 15 values of $\beta$, step 2 was repeated 100 times, and the average squared error was calculated.

4. Steps 1 through 3 were repeated 100 times (at each time, 15 new $\beta$ values were generated in step 1). The mean of the average squared errors of each estimator from the true reliability stored in step 3 for each of the 100 repetitions was computed, giving an estimated mean squared error (MSE).

The above procedures were repeated for several different values of $\gamma$, $\alpha(R)$, and $\lambda$. Some of the results of the simulations are given in Tables 1 and 2. The tables give the average true values of reliability and the MSE's of the three sequences of estimators at each of the 15 stages.

The results indicate that the estimated mean squared errors of G are generally smaller than those of $G^*$ at each stage when the sample sizes are equal. Also, for each of the estimators, the mean squared errors for sample size 10 are smaller than those for sizes 3 and 5. This, however, follows from the observation that $p_m \to 0$ as $m \to \infty$. Also, G and H perform equally well in the sense that neither of the MSE's of G or H is uniformly smaller than the other throughout the 15 stages when sample sizes are unequal.

Hence, nothing can be said definitely about which estimator is generally better than either of the other two for small n. Obviously, the Korwar-Hollander estimators G perform better in the sense of smaller asymptotic risk than $G^*$, although for unequal sample sizes G and H are very close. In addition it was observed that the choice of the value of $\alpha(R)$ had little effect on the results after the first few stages of the process.

Table 1

Comparison of Three Sequences of Estimators

$\gamma = 1,\ \alpha(R) = 0.5,\ \lambda = 4,\ t = 0.2$

| Stage | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **True R (.2)** | | .33 | .33 | .35 | .38 | .32 | .35 | .34 | .32 | .31 | .34 | .34 | .34 | .38 | .35 | .30 |
| **Sample Size** | | MSE | | | | | | | | | | | | | | |
| G* | m = 3 | .469 | .429 | .409 | .440 | .386 | .404 | .409 | .406 | .388 | .434 | .394 | .411 | .429 | .415 | .366 |
| | m = 5 | .276 | .271 | .263 | .289 | .262 | .277 | .275 | .260 | .259 | .287 | .259 | .269 | .275 | .270 | .233 |
| | m = 10 | .142 | .141 | .141 | .160 | .141 | .144 | .141 | .138 | .132 | .159 | .135 | .147 | .156 | .142 | .126 |
| G (=H) | m = 3 | .469 | .429 | .403 | .433 | .372 | .387 | .398 | .392 | .378 | .418 | .376 | .396 | .415 | .399 | .349 |
| | m = 5 | .276 | .271 | .259 | .286 | .257 | .271 | .270 | .253 | .255 | .282 | .252 | .263 | .269 | .263 | .228 |
| | m = 10 | .142 | .141 | .140 | .159 | .139 | .143 | .140 | .137 | .130 | .157 | .134 | .146 | .155 | .140 | .125 |

Table 2

Comparison of Three Sequences of Estimators

$\gamma = 1,\ \alpha(R) = 4,\ \lambda = 4,\ t = 0.2$

| Stage | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| True R(.2) | | .33 | .38 | .35 | .32 | .34 | .34 | .35 | .31 | .34 | .34 | .31 | .31 | .34 | .33 | .33 |
| Sample Sizes | | 2 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 | 10 | 11 | 11 | 12 | 12 | 12 |
| MSE | $G^*$ | .709 | 1.11 | .476 | .292 | .256 | .230 | .207 | .188 | .194 | .161 | .143 | .149 | .160 | .151 | .139 |
| | G | .709 | 1.11 | .467 | .249 | .209 | .184 | .167 | .149 | .145 | .137 | .124 | .117 | .122 | .117 | .114 |
| | H | .709 | 1.11 | .467 | .248 | .210 | .185 | .165 | .151 | .149 | .136 | .123 | .118 | .124 | .118 | .114 |
| Sample Sizes | | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 5 | 6 | 6 | 6 | 7 | 7 | 7 |
| MSE | $G^*$ | .711 | 1.11 | .454 | .480 | .382 | .316 | .324 | .296 | .299 | .311 | .236 | .255 | .261 | .258 | .223 |
| | G | .711 | 1.11 | .431 | .440 | .362 | .280 | .276 | .255 | .241 | .227 | .200 | .196 | .193 | .197 | .180 |
| | H | .711 | 1.11 | .431 | .446 | .360 | .278 | .275 | .252 | .237 | .233 | .200 | .198 | .196 | .196 | .181 |

## REFERENCES

Canavos, G. S. and C. P. Tsokos. Bayesian estimation of life parameters in the Weibull distribution. Operations Research 21 (1973), 755-763.

Ferguson, T. S. A Bayesian analysis of some nonparametric problems. Ann. Statist. 1 (1973), 209-230.

Ferguson, T. S. Prior distributions on spaces of probability measures. Ann. Statist. 2 (1974), 615-629.

Korwar, R. M. and M. Hollander. Empirical Bayes estimation of a distribution function. Ann. Statist. 4 (1976), 581-588.

Korwar, R. M. and M. Hollander. Nonparametric estimation of distribution functions. The Theory and Applications of Reliability, Vol. 1, ed. by C. P. Tsokos and I. N. Shimi, Academic Press, New York, 1977, 85-107.